MATHEMATICS

Paper : MATC1.1

Full Marks : 80 Time : 4 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols have their usual meanings.

GROUP–A

(Real Analysis-I)

(Marks : 25)

Answer **Q. No. 1** and any **two** from the rest:

1. Answer **(a)** or **(b)**:

a) Show that if $f:[a, b] \to \mathbb{R}$ is a function of bounded variation on [a, b], then f is bounded on [a, b]. If f and g are two functions of bounded variation on [a, b], then show that f.g is also of bounded variation on[a, b]. 2+3

b) Show that every absolute continuous function defined over [a, b] is of bounded variation. Show that

[Turn over]

i) $a+a=a$ and

ii) $n+a=a$,

where n is any finite cardinal and 'a' is denumerable cardinal. 3+1+1

2. a) Let f be a function of bounded variation on [a, b]. If f is continuous at a point $x_0 \in [a, b]$, then show that the function $F(x) = V_f[a, x]$ is also continuous at x_0. 3

b) Show that if f is absolute continuous on [a, b], then so is $|f|$. Hence or otherwise verify whether the following function is absolutely continuous on [0, 1].

$$f(x) = x^2\left|\sin\frac{1}{x}\right|, \quad x \neq 0$$
$$= 0 \quad , \quad x = 0.$$
2+2

c) Given the cardinal numbers m and n, show that $m \leq n$ if and only if there is a cardinal number p such that $n=m+p$. 3

3. a) For any set A, show that $\overline{\overline{P(A)}} = 2^{\overline{\overline{A}}}$. 4

b) If m, n, p are the cardinal numbers with $m \leq n$, then prove that $m^p \leq n^p$. 1

c) State and prove Causin's Lemma. 1+4

4. a) Define Cantor set. $1\frac{1}{2}$

b) If f(x) and $\phi(x)$ are bounded functions on [a, b] and $\phi(x)$ is non-decreasing on [a, b], then obtain a necessary condition that the integral

$$\int_a^b f(x)d\phi(x) \text{ should exist.}$$

Let the function f(x) and $\phi(x)$ defined on [–1, 1] be given by

$$f(x)=\begin{cases}0, & -1\leq x\leq 0\\ 1, & 0<x\leq 1\end{cases} \text{ and } \phi(x)=\begin{cases}0, & -1\leq x<0.\\ 1, & 0\leq x\leq 1\end{cases}$$

Then show that $\int_{-1}^{1} f(x)d\phi(x)$ does not exist.

$1\frac{1}{2}+2$

c) State and prove the theorem for integration by parts in connection with Riemann Stieltjes integral. 5

GROUP–B

(Complex Analysis–I)

(Marks : 30)

Answer any **three** questions:

5. a) If a function f is differentiable at z_0, then show that it is continuous at z_0. Is the converse true? Give reason in support of your answer.

   b) If f(z)=u(x, y)+iv(x, y) is analytic in a domain D and |f(z)| is constant in D, then show that f(z) is constant in D. 6+4

6. a) If $f(z)=\frac{1}{z^2}$ and Γ be the straight line joining the points i and 2+i, then show that

   $$\left|\int_\Gamma f(z)dz\right| \le 2.$$

   b) Evaluate $\oint_C \frac{dz}{(z-\alpha)^m}$, where m is an integer and C is the circle $|z-\alpha| = r$.

   c) Evaluate $\oint_{|z|=1} \frac{dz}{z+2}$ and deduce that

   $$\int_0^\pi \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0.$$ 2+3+5

7. a) State and prove Cauchy's integral formula.

b) If f is analytic within and on a circle $C:|z-\alpha|=r$ and $|f(z)|\leq M$ for all z on C, where M is a positive constant, then show that

$$|f^{(n)}(\alpha)|\leq\frac{Mn!}{r^n} \text{ for } n=0, 1, 2, ...$$ 7+3

8. a) State and prove Liouville's theorem.

b) Given a rectifiable curve L, suppose that the series $f(z)=\sum_{n=1}^{\alpha} f_n(z)$ is uniformly convergent on L and every term $f_n(z)$ is continuous on L. Prove that

$$\int_L f(z)dz=\sum_{n=1}^{\infty}\int_L f_n(z)dz.$$ 6+4

9. State and prove Taylor's theorem for analytic functions. 10

GROUP–C

(Functional Analysis-I)

(Marks : 25)

Answer **Q. No. 10** and any **two** from the rest:

10. Answer any **one** question: $5 \times 1 = 5$

a) i) Using contraction principle, check whether the following system of equations possesses a unique solution or not.

$$15x=3y-5z+18$$

$$18y=-2x+3z-16$$

$$20z=4x-5y+7.$$

ii) Prove that any contraction on a metric space to itself is uniformly continuous.

b) Stating the conditions prove that Fredholm integral equation of second kind has a unique continuous solution.

11. a) Prove that the metric space C[a, b] with the sub metric is complete. Also prove that convergence in C[a, b] is equivalent to uniformly convergence of functions on [a, b].

b) If B_0 be an open ball in a metric space (X, d) and if E be a subset of X which is not dense in B_0, then for every $\varepsilon > 0$, prove that there is an open ball B in X such that $\overline{B} \subset B_0$, $\overline{B} \cap E = \phi$ and $d(\overline{B}) < \varepsilon$. (4+2)+4

12. a) Does every closed and bounded subset of any dimensional normed linear space compact? Clarify your answer with proper justifications.

b) Is it possible to make every finite dimensional linear space to a Banach space? Justify.

7+3

13. a) State and prove Riesz's Lemma.

b) Prove that the metric space l_p is a Banach space. (2+4)+4

MATHEMATICS

Paper : MATC-1.2

(Ordinary Differential Equations, Partial Differential Equations)

Full Marks : 80 Time : 4 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols have their usual meanings.

GROUP–A

(Ordinary Differential Equations)

(Marks : 40)

Answer any **four** questions: 10×4=40

1. a) Define ordinary point and regular singular point of a second order linear differential equation in complex domain.

   b) State and Prove Fuch's theorem. 2+8

2. a) Obtain a solution of the differential equation

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+\left(z^2-\lambda^2\right)w=0$$

near z=0, λ is neither zero nor an integer.

[Turn over]

b) State and Prove Sturm Comparison Theorem.

6+4

3. a) Find the eigen values and eigen functions of the differential equation

$$\frac{d^2y}{dx^2}+\lambda y=0$$

satisfying the boundary conditions y(0)=0 and $y\left(\frac{\pi}{2}\right)=0$.

b) Prove that for an integral values of n, $J_{-n}(z)=(-1)^n J_n(z)$ where $J_n(z)$ is the Bessel's function of first kind.

c) Find the adjoint equation of the differential equation

$$\left(1+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0.$$ 4+3+3

4. a) Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous differential equation

$$y^{(n)}+p_1(x)y^{(n-1)}+\ldots+p_n(x)y=0,$$

where $p_1(x), p_2(x), \ldots, p_n(x)$ are complex valued functions on an interval I=[a, b]. Show that the solutions are linearly independent if and only if the Wronskian $W(y_1, y_2, \ldots, y_n)(x) \neq 0 \ \forall x \in I$.

b) Solve the following differential equation by reducing to normal form:

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(a^2 + \frac{2}{x^2}\right)y = 0.$$ 6+4

5. a) State and Prove Gronwall's Lemma.

b) Using Green's function, solve the boundary value problem

$$\frac{d^2u}{dx^2} - u + 1 = 0, \ 0 < x < 1,$$

$$u(0) = u(1) = 0.$$ 6+4

6. a) Determine the interval of existence of solutions of the initial value problem

$$\frac{dy}{dx} = y^2 + \cos^2 x, \ y(0) = 0$$

in the region R given by

$$R : \left\{(x, y) : 0 \leq x \leq a, \ |y| \leq b, \ a > \frac{1}{2}, \ b > 0\right\}.$$

b) Show that

$$\frac{d}{dz}F(a, b; c; z) = \frac{ab}{c}F(a+1, b+1; c+1; z)$$

where F(a, b; c; z) is the hypergeometric series.

c) Using the method of variation of parameters, find a particular solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 64xe^{-x}.$$ 3+3+4

GROUP–B

(Partial Differential Equations)

(Marks : 40)

Answer any **four** questions: 10×4=40

7. a) Solve the initial value problem for the quasi-linear equation $zz_x + z_y = 1$ containing the initial data curve $x_0 = s$, $y_0 = s$, $z_0 = \frac{1}{2}s$ for $0 \le s \le 1$.

b) Solve

$$\left(D^2 - 3DD' + 2D'^2 - D + 2D'\right)z = (2+4x)e^{-y}.$$

c) Find $\frac{1}{D^2 + 3DD' + 2D'^2}(x+y)$. 6+3+1

8. a) Find the characteristics of the equation $pq = x$ and determine the integral surface which passes through the curve

$$y = \frac{1}{2},\ z = x.$$

b) Solve $\left(x^2D^2 + 2xyDD' + y^2D'^2\right)z = x^3y^2$.

6+4

9. a) Reduce the following partial differential equation to its canonical form and indicate the nature of the equation

$$Z_{xx} - 2\sin x\, Z_{xy} - \cos^2 x\, Z_{yy} - \cos x\, Z_y = 0.$$

b) Suppose that u(x, y) is harmonic in a bounded domain D and continuous in $\bar{D} = D \cup B$. Prove that u attains its maximum on the boundary B of D.

6+4

10. Obtain the solution of the problem

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2},\ k > 0,\ t > 0,\ a < x < b$$

under the conditions

i) $u(a, t) = u(b, t) = 0,\ t \geq 0$

ii) u(x, t) is finite as $t \to \infty$

iii) $u(x, 0) = f(x),\ a \leq x \leq b$

iv) $f(a) = f(b) = 0$.

10

[Turn over]

11. a) Let $G(r, r')$ denotes the Green's function for the Dirichlet problem involving Laplace operator. Show that $G(r, r')$ has the symmetric property.

b) Let u be a classical solution to wave equation $u_{tt} - u_{xx} = 0$ for $(x, t) \in \mathbb{R} X (0, \infty)$. Show that

i) for each $y \in \mathbb{R}$, the function $\omega(x, t) := u(x - y, t)$ is also a solution of the wave equation.

ii) for each $k \in \mathbb{N}$, the function

$$\omega(x, t) := \frac{\partial^k u}{\partial x^k}(x, t)$$

is also a solution of the wave equation.

iii) for each $a > 0$, the function $\omega(x, t) := u(ax, at)$ is also a solution of the wave equation.

c) State the wellposedness result for the cauchy problem of wave equation in one space dimension with appropriate assumptions on the Cauchy data.

5+3+2

12. a) Solve the Cauchy problem for the non-homogeneous wave equation

$$u_{tt} - u_{xx} = t^7 \text{ for } x \in \mathbb{R},\ t > 0$$

$$u(x, 0) = 2x + \sin x, \text{ for } x \in \mathbb{R},$$

$$u_t(x, 0) = 0 \text{ for } x \in \mathbb{R}.$$

b) Solve the following initial boundary value problem using separation of variables and state the suitable assumptions on ϕ and ψ.

$$u_{tt} - c^2 u_{xx} = 0 \text{ for } 0 \le x \le l,\ t > 0$$

$$u(x, 0) = \phi(x) \text{ for } 0 \le x \le l,$$

$$u_t(x, 0) = 0 \text{ for } 0 \le x \le l$$

$$u(0, t) = 0 \text{ for } t \ge 0$$

$$u(l, t) = 0 \text{ for } t \ge 0.$$

4+6

MATHEMATICS

Paper : MATC-1.3

(Mechanics-I, Abstract Algebra-I, Operations Research-I)

Full Marks : 80 Time : 4 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols have their usual meanings.

GROUP–A

[Mechanics-I (Potential Theory)]

(Marks : 30)

Answer any **three** questions: 10×3=30

1. Define potential U at any point P(x, y, z) due to a double layer. Prove that the potential of a double layer on a surface S has a jump value on passing through S at a point in the direction of the normal at that point. 4+6

2. State and Prove Gauss's Integral theorem. Show that the average value over the surface of a sphere of the potential U of masses lying entirely outside the

[Turn over]

sphere is equal to the value of the potential at the centre of the sphere. 2+3+5

3. Define Harmonic function. Express harmonic function of any interior point of the region in terms of Green's function. Also express the properties to be satisfied by the Green's function. 2+5+3

4. Define solid and surface spherical harmonics. If $H_n(x, y, z)$ be a solid spherical harmonic of degree n, homogeneous in x, y, z then prove that

$$\nabla^2\left(r^m H_n\right) = m(m+2n+1)r^{m-2}H_n,$$

where $r^2 = x^2+y^2+z^2$. 2+2+6

5. Define potential of an attracted particle. Prove that force components X, Y, Z of a volume distribution of matter of piecewise continuous density χ throughout a regular region are continuous in the volume V. Deduce greens second identity.

2+5+3

GROUP–B

(Abstract Algebra-I)

(Marks : 25)

6. Answer any **five** questions: 5×5

a) Let G be a finite group of order n such that n is divisible by a prime p. Then show that G contains a subgroup of order p. 5

b) Define a simple group. Prove that no group of order 36 is simple. 1+4

c) State and Prove Sylow's First theorem. 1+4

d) Distinguish between internal direct product and external direct product of groups. Find the number of elements of order 5 in $Z_{15} \times Z_5$. Also show that the direct product z×z is not a cyclic group. 2+2+1

e) Define a solvable group with an example. Prove that every subgroup of a solvable group is solvable. 1+1+3

f) Obtain a necessary and sufficient condition for a group to be solvable. Hence check the solvability of S_6. 4+1

g) For each prime p, show that a finite group G has a Sylow p-subgroup. Is every finite p-group solvable? Justify your answer. 3+2

[Turn over]

h) Prove that every group of order 35 is cyclic. Write down the class equation for S_3. 3+2

GROUP–C

(Operations Research-I)

(Marks : 25)

Answer **Q. No. 7** and any **one** from the rest.

7. Discuss the "Gomory's cutting plane constraint". Solve the following integer programming problem

Maximize $Z = x_1 + x_2$

subject to $3x_1 + 2x_2 \le 5$

$x_2 \le 2$

$x_1, x_2 \ge 0$ and are integers.

5+8=13

OR

State the Kuhn-Tucker necessary conditions for the problem

Maximize Z = f(x)

subject to gi(x)(<0, i = 1, 2...m)

where $x = (x_1, x_2, ..., x_n)^T$

Also prove that Kuhn-Tucker necessary conditions are also sufficient conditions if f(x) is concave and all gi(x) are convex function of x. Determine x_1 and x_2 so as to

Maximize $Z=12x_1+21x_2+2x_1x_2-2x_1^2-2x_2^2$

subject to $x_2 \le 8$

$x_1 + x_2 \le 10$

$x_1, x_2 \ge 0$. 2+4+7=13

8. Use Dual-Simplex method to solve the L.P.P.

Minimize $Z=2x_1+3x_2+x_3$

subject to $2x_1 + x_2 + 2x_3 \ge 3$

$3x_1 + 2x_2 + x_3 \ge 4$

$x_1, x_2, x_3 \ge 0$.

State the advantages of Revised Simplex method.

10+2=12

9. Describe "Johnson's Procedure" for processing n jobs through two machines, M_1 and M_2 in the order M_1M_2. A book binder has one printing press, one binding machine and manuscripts of 7 different books. The times required for performing printing and binding operations for different books are shown below:

Book	1	2	3	4	5	6	7
Printing time (hours):	20	90	80	20	120	15	65
Binding time (hours):	25	60	75	30	90	35	50

Determine the optimum sequence of processing of books in order to minimize the total time required to bring out all the books. 6+6=12

MATHEMATICS

Mechanics of Solids, Non-Linear Dynamics

Paper : MATA1.4

[Applied Stream]

Full Marks : 80 Time : 4 Hours

The figures in the right-hand margin indicate marks.

Symbols have their usual meanings.

GROUP–A

(Mechanics of Solids)

(Marks : 40)

Answer any **four** questions: 10×4=40

1. a) Write down Lagrangian and eulerian strain tensors for finite deformation of a continuous medium and show that there is no distinction between these two strain components in case of infinitesimal deformation. 5

   b) At a point, the strain tensor is given by

$$(e_{ij}) = \begin{pmatrix} 1 & 3 & -2 \\ 3 & 1 & -2 \\ -2 & -2 & 6 \end{pmatrix}.$$

[Turn over]

Determine the principal strains and the principal directions of strain. Also find maximum value of normal strain and shearing strain. 5

2. a) Deduce the compatibility equations for infinitesimal strain components in an elastic medium occupying a simply connected region. 5

b) Under what conditions the following is a possible system of strain in an elastic body?

$$e_{xx} = a + b\left(x^2 + y^2\right) + x^4 + y^4$$

$$e_{yy} = \alpha + \beta\left(x^2 + y^2\right) + x^4 + y^4$$

$$e_{xy} = A + B_{xy}\left(x^2 + y^2 - C^2\right)$$

$$e_{zz} = e_{yz} = e_{zx} = 0$$

a, b, α, β, A, B, C are all constants. 5

3. a) Discuss Mohr's circle for three dimensional state of stress to determine

i) The magnitude and orientation of the principal stresses and

ii) the magnitude and orientation of the maximum shearing stresses and associated normal stresses. 5

b) Stress ~~sor of a point P is given by

$$(Tij) = \begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$

Determine stress vector on the plane of P whose unit normal is $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$. 5

4. a) What is strain energy function? Write down its form for an isotropic medium and obtain its expression in terms of strain invariants.

b) What are 'elastic moduli'? Define Young's modulus E and Poisson's ration σ. Show that

$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$, $\sigma = \frac{\lambda}{2(\lambda + \mu)}$, where λ, μ are Lamé constants. 5+5

5. a) Deduce Beltrami-Michell compatibility equations for stresses.

b) Show that the following stress components are not the solutions of the problem in elasticity, even though they satisfy the equations of equilibrium with zero body forces:

$\tau_{11} = \alpha\left[x_2^2 + \sigma\left(x_1^2 - x_2^2\right)\right]$, $\tau_{12} = -2\alpha\sigma x_1 x_2$,

$\tau_{13} = 0$, $\tau_{22} = \alpha\left[x_1^2 + \sigma\left(x_2^2 - x_1^2\right)\right]$, $\tau_{23} = 0$,

$\tau_{33} = \alpha\sigma\left(x_1^2 + x_2^2\right)$. 5+5

6. a) Show that there are two possible plane waves propagating in an unbounded, isotropic and homogeneous elastic medium. Which of the two waves propagates with higher velocity? What is the nature of these two waves?

b) What is plane stress? Explain how Airy's stress function can be used to determine the stress distribution in an elastic body in case of plane stress and in the absence of body forces.

5+5

GROUP–B

(Non-linear Dynamics)

(Marks : 40)

Answer any **four** questions: 10×4=40

7. Define asymptotic stability. State second or direct method of Lyapunov. Use it to verify whether the following system is asymptotically stable:

$$\frac{dx}{dt} = -2x + yz - x^2$$

$$\frac{dy}{dt} = y - xz - y^3$$

$$\frac{dz}{dt} = xy - y^2$$

2+3+5

8. State and prove Bendixson's negative criterion. Check the stability of the following system at (0, 0).

$$\dot{x} = -4y - x^3$$

$$\dot{y} = 3x - y^3$$

Also find the Lyapunov function of the above system of equations and use it to prove that the system is asymptotically stable. 1+2+2+2+3

9. Consider the system:

$$\frac{dx}{dt} = r_1 x - a_{11}x^2 - a_{12}xy$$

$$\frac{dy}{dt} = r_2 y - a_{21}xy - a_{22}y^2$$

where $r_1, r_2, a_{11}, a_{12}, a_{21}, a_{22} > 0$.

Find the planar equilibrium and check the stability of the system about that equilibrium. What is hyperbolic equilibrium point of a system of differential equations? State Hartman-Grobman theorem. 3+4+1+2

10. State Routh-Hurwitz criterion. Consider the system $\frac{dx}{dt} = rx - \frac{axy}{k+x}, \frac{dy}{dt} = \frac{bxy}{k+x} - dy$. where r, k, a, b, d>0.

Find the planar equilibrium and check the stability of the system. 3+3+4

11. a) Define 'flow evolution operator' of a dynamical system.

b) Find the evolution operator of the dynamical system $\dot{x} = -x^2$. Show that the operator forms a commutative dynamical group.

c) Find the fixed points and using the concept of flow analysis, analyze the local stability of the fixed points of $\dot{x} = x - x^3$. 2+4+4

12. a) What do you mean by 'bifurcation' of a dynamical system?

b) Discuss transcritical bifurcation of the system $\dot{x} = \mu x - x^2$, $x \in R$ with $\mu \in R$ as a parameter.

c) Obtain the relation between 'a' and 'b', so that the system $\dot{x} = x(1 - x^2) - a(1 - e^{-bx})$ undergoes a transcritical bifurcation at x = 0.

1+5+4

MATHEMATICS

Paper : MATP-1.4 (Pure)

(Differential Geometry-I, Topology-I [Pure Stream])

Full Marks : 80 Time : 4 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols have their usual meanings.

Write the answers to questions of each group in separate books.

Pure Stream

GROUP–A

(Differential Geometry-I)

(Marks : 40)

Answer any **four** questions: $10\times4=40$

1. a) Define a unit speed curve. Show that if γ is a unit speed curve, then $\ddot{\gamma}$ is either zero or perpendicular to $\dot{\gamma}$.

   b) Define arc length function of a curve γ. Find the arc length function of the curve

$$\gamma(t)=\left(e^{t}\cos t,\ e^{t}\sin t\right).$$

[Turn over]

c) Define reparametrization of a parametrized curve with example.

d) Find the curvature of the curve

$$\gamma(\theta) = (a\cos\theta,\ a\sin\theta,\ b\theta),$$

$$-\infty < \theta < \infty$$

where a and b are constants. 2+3+2+3

2. a) Let $\gamma(s)$ be a unit speed plane curve and let $\phi(s)$ be the angle through which a fixed unit vector must be rotated anti-clockwise to bring it into coincidence with the unit tangent vector t of γ. If k_s denotes the signed curvature of the curve, show that

$$k_s = \frac{d\phi}{ds}.$$

b) Let $k:(\alpha, \beta) \to R$ be any smooth function. Then, show that there exists a unit speed curve $\gamma:(\alpha, \beta) \to R^2$ whose signed curvature is k. Further, show that, if $\tilde{\gamma}:(\alpha, \beta) \to R^2$ is any other unit speed curve whose signed curvature is k, there is a rigid motion M of R^2 such that $\tilde{\gamma}(s) = M(\gamma(s))$ for $s \in (\alpha, \beta)$. 3+7

3. a) Deduce Serret-Frenet formulae for a parametrized curve.

b) Compute the torsion of the curve

$$\gamma(\theta) = (a\cos\theta,\ a\sin\theta,\ b\theta),$$

where $-\infty < \theta < \infty$ and a, b are constants.

c) Let γ be a regular curve in R^3 with nowhere vanishing curvature. Then show that the image of γ is contained in a plane if and only if the torsion is zero at every point. 5+2+3

4. a) Let (aij) be a skew symmetric 3×3 matrix. Let v_1, v_2 and v_3 be smooth vector valued functions of a parameters satisfying the differential equations

$$\dot{v}_i = \sum_{j=1}^{3} a_{ij} v_j,$$

for i=1, 2, and 3, and suppose that for some parameter value s_0 the vectors $v_1(s_0)$, $v_2(s_0)$, and $v_3(s_0)$ are orthonormal. Show that the vectors $v_1(s)$, $v_2(s)$ and $v_3(s)$ are orthonormal for all values of s.

[Turn over]

b) Let $f : \mathbb{R}^2 \to R$ be given by

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & , \ (x, y) \neq (0, 0) \\ 0 & , \ (x, y) = (0, 0). \end{cases}$$

Verify whether all the directional derivatives of f exists at (0, 0). Is f differentiable at (0, 0)? 5+5

5. a) Verify whether the curve

$$\gamma(t) = \left(\frac{1+t^2}{t}, t+1, \frac{1-t}{t} \right) \text{ is planar.}$$

b) Describe all curves in R^3 which have constant curvature $k > 0$ and constant torsion τ.

c) Show that the torsion of a regular curve is a smooth function.

d) Let γ be a unit speed curve in R^3 with constant curvature and zero torsion. Show that γ is part of a circle. 2+3+2+3

6. State and prove isoperimetric inequality for simple closed curve. 10

GROUP–B

(Topology-I)

(Marks : 40)

Answer any **four** questions: 10×4=40

7. a) Prove that the 1st projection map $\pi_1 : X \times Y \to X$ is an open map, that it maps open sets into open sets.

   b) Prove that $\{(a, b)\times(c, d): a, b, c, d \in \mathbb{Q}\}$ is a base for $\mathbb{R}^2$.

   c) Consider $\mathbb{R}_l^2$ and $\mathbb{R}_u^2$. Consider the Line $L = \{(x, -x) : x \in \mathbb{R}\}$. Then L is a subset of both $\mathbb{R}_l^2$ and $\mathbb{R}_u^2$. Prove that subspaces topologies on L from both $\mathbb{R}_l^2$ and $\mathbb{R}_u^2$ are same.

   d) Let Y be a subspace of X. If $A \subset Y$ prove that $cl_Y A = Y \cap cl_X A$. 2+2+3+3

8. a) Prove that product of any two Hausdorff spaces is Hausdorff.

   b) Give an example of a continuous bijection which is not a homeomorphism.

   c) Consider identity map from $\mathbb{R}_u$ to R_l and that form $\mathbb{R}_l$ to R_u. Which one is continuous and which one is discontinuous? 4+3+3

9. a) Let $\{X_\alpha\}$ be an indexed family of spaces and $A_\alpha \subset X_\alpha$. Prove that

$$\prod \overline{A_\alpha} = \overline{\prod A_\alpha}.$$

b) Let $f : A \to \Pi_{\alpha\in I} X_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha\in I},$$

where $f_\alpha : A \to X_\alpha$ for each α. Then prove that f is continuous iff each f_α is continuous.

c) Given sequences $(a, a_2, \ldots)$ and $(b, b_2, \ldots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the equation

$$h((x_1, x_2, \ldots)) = (a_1 x + b_1, a_2 x + b_2, \ldots).$$

3+4+3

10. a) Prove that every regular 2nd countable space is normal.

b) Is every normal space is 2nd countable, justify your answer.

c) Prove that continuous image of every separable and Lindeloff space are so.

4+3+3

11. a) Show that if X is normal, every pair of disjoint closed sets have neighbourhood whose closures are disjoint.

b) Let $f : X \to \mathbb{R}$ be a continuous function. Then prove that the zero set $Z(f) = \{x \in X : f(x) = 0\}$ of f is a closed set. Prove that in a metric space every closed set is a zero set.

c) Show that product of normal spaces may not be normal. 3+4+3

12. a) Prove that complete regularity is both heridity and productive property.

b) Give an example of a completely regular space which is nor normal.

c) Prove that if X is a normal space and A be a closed G_δ set in X, i.e. countable intersection of open sets, then A is a zero set. 4+3+3

———